

The quantum H_3 integrable system

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Abstract

The quantum H_3 integrable system is a $3D$ system with rational potential related to the non-crystallographic root system H_3 . It is shown that the gauge-rotated H_3 Hamiltonian as well as one of the integrals, when written in terms of the invariants of the Coxeter group H_3 , is in algebraic form: it has polynomial coefficients in front of derivatives. The Hamiltonian has infinitely-many finite-dimensional invariant subspaces in polynomials, they form the infinite flag with the characteristic vector $\vec{\alpha} = (1, 2, 3)$. One among possible integrals is found (of the second order) as well as its algebraic form. A hidden algebra of the H_3 Hamiltonian is determined. It is an infinite-dimensional, finitely-generated algebra of differential operators possessing finite-dimensional representations characterized by a generalized Gauss decomposition property. A quasi-exactly-solvable integrable generalization of the model is obtained. A discrete integrable model on the uniform lattice in a space of H_3 -invariants "polynomially"-isospectral to the quantum H_3 model is defined.

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I. INTRODUCTION

About 30 years ago, Olshanetsky and Perelomov developed the Hamiltonian Reduction Method, later known as the Projection Method (for a review, see [1]). This method provides an opportunity to construct on a regular basis the non-trivial multidimensional quantum (and classical) Hamiltonians, which are associated to the crystallographic root spaces of the classical (A_N, B_N, C_N, D_N) and exceptional $(G_2, F_4, E_{6,7,8})$ Lie algebras. All these systems are symmetric with respect to the corresponding Weyl group transformations. The Olshanetsky–Perelomov Hamiltonians have the property of complete integrability (the number of integrals of motion in involution is equal to the dimension of the configuration space). There are three types of the Hamiltonians with rational, trigonometric and elliptic potentials, respectively. The Hamiltonians with rational and trigonometric potentials are exactly solvable (the spectrum can be found explicitly, in a form of a first- or second-degree polynomial in the quantum numbers, respectively).

A Hamiltonian with rational potential associated to a Lie algebra g of rank N with root space Δ has a form

$$\mathcal{H}_\Delta = \frac{1}{2} \sum_{k=1}^N \left[-\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2 \right] + \frac{1}{2} \sum_{\alpha \in \mathcal{R}^+} g_{|\alpha|} \frac{|\alpha|^2}{(\alpha \cdot x)^2}, \quad (1)$$

where $R_+ \in \Delta$ is the set of positive roots, $\omega \in \mathbb{R}^+$ is a real parameter, $g_{|\alpha|} = \nu_{|\alpha|}(\nu_{|\alpha|} - 1)$ are coupling constants depending only on the root length, and $x = (x_1, x_2, \dots, x_N)$ is the coordinate vector. The configuration space is the principal Weyl chamber of the root space (see [1]). The ground state eigenfunction and its eigenvalue are given by

$$\Psi_0(x) = \left(\prod_{\alpha \in R_+} (\alpha \cdot x)^{\nu_{|\alpha|}} \right) \exp \left(-\frac{\omega}{2} t_2^{(\Omega)} \right) \quad (2)$$

and its eigenvalue

$$E_0 = \left(\frac{N}{2} + \sum_{\alpha \in \mathcal{R}^+} \nu_{|\alpha|} \right) \omega, \quad (3)$$

where $t_2^{(\Omega)}$ is the invariant of the degree two (for definition see below). It is indicated in [1] that the Hamiltonian (1) with the property (2)-(3) can be introduced for the non-crystallographic root systems H_3 , H_4 and dihedral $I_2(m)$ with a Coxeter group as a symmetry of the system. All that is not true for the H_3 , H_4 and $I_2(m)$ Hamiltonians with trigonometric or elliptic potential. The complete integrability of (1) for the case of non-crystallographic root systems has been proven in [2] using the formalism of quantum Lax pairs.

Following [3], we make three definitions.

DEFINITION 1. A multivariate linear differential operator is said to be in algebraic form if its coefficients are polynomials in the independent variable(s). It is called algebraic if by an appropriate change of the independent variable(s), it can be written in an algebraic form.

DEFINITION 2. Consider a finite-dimensional (linear) space of multivariate polynomials defined as a linear space spanned in the following way:

$$P_n^{(\vec{\alpha})} = \langle x_1^{p_1} x_2^{p_2} \dots x_d^{p_d} | 0 \leq \alpha_1 p_1 + \alpha_2 p_2 + \dots + \alpha_d p_d \leq n \rangle ,$$

where the α 's are positive integers and $n \in \mathbb{N}$. It represents the Newton polytope in a form of a rectangular pyramid. Its *characteristic vector* is the d -dimensional vector with components α_i :

$$\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d) . \quad (4)$$

For some characteristic vectors, the corresponding polynomial spaces may have a Lie-algebraic interpretation, in that they are the finite-dimensional representation spaces for some Lie algebra of differential operators. The smallest characteristic vector is $\vec{\alpha}_0 = (1, 1, \dots, 1)$. It corresponds to the finite-dimensional representation space for the Lie algebra $gl(d+1)$ of the first order differential operators acting in \mathbf{R}^d [4]. We will call such a space the *basic* space as well as the associated flag will be called the *basic* flag.

DEFINITION 3. Take the infinite set of spaces of multivariate polynomials $P_n \equiv P_n^{(\vec{\alpha})}$, $n \in \mathbb{N}$, defined as above, and order them by inclusion:

$$P_0 \subset P_1 \subset P_2 \subset \dots \subset P_n \subset \dots .$$

Such an object is called an *infinite flag (or filtration)*, and is denoted $P^{(\vec{\alpha})}$. If a linear differential operator preserves such an infinite flag, it is said to be *exactly-solvable*. It is evident that every such operator is algebraic (see [4]). If the spaces P_n can be viewed as the finite-dimensional representation spaces of some (Lie) algebra g , then g is called the *hidden algebra* of the exactly-solvable operator.

If a linear operator preserves several flags and among them there is a flag for which $\dim P_n$ is *maximal* for any given n , such a flag is called *minimal*. Every flag can be characterized by a normal vector $\mathbf{n}_{\vec{\alpha}}$ to the hyperplane $\alpha_1 p_1 + \alpha_2 p_2 + \dots + \alpha_d p_d = n$ of the base of the

Newton polytope. This normal vector is, in fact, the characteristic vector. It is clear that for minimal characteristic vector the angle with basic characteristic vector $(\vec{\alpha}, \vec{\alpha}_0)$ is minimal.

For any root system Δ there exist $N = \text{rank}(\Delta)$ homogeneous, algebraically independent polynomials which are invariant with respect to the Coxeter group. They are called the invariants. The lowest possible degrees a of these invariants are the *degrees* of the Coxeter group. Each invariant is defined ambiguously, up to a non-linear combination of the invariants of the lower degrees.

One of the ways to find an invariant of degree a (denoted as $t_a^{(\Omega)}$) is to make averaging over an orbit Ω ,

$$t_a^{(\Omega)}(x) = \sum_{w \in \Omega} (w \cdot x)^a, \quad (5)$$

(see e.g. [5]), where x 's are some formal variables which can be identified with the Cartesian coordinates. It is worth mentioning that for any Coxeter group there exists a second degree invariant $t_2^{(\Omega)}$, this invariant does not depend on the chosen orbit. Later on we will use the invariants (5) as new variables. We will call them the *orbit variables*.

For all the crystallographic root systems algebraic representations of all quantum Hamiltonians, both rational and trigonometric, have been found ([6]-[12]). The general strategy which was used to find a minimal flag for the rational Hamiltonians is the following: (i) as a first step we consider the similarity-transformed version of (1), namely $h \propto \Psi_0^{-1}(\mathcal{H} - E_0)\Psi_0$, (ii) then, we choose a certain orbit to construct a particular set of variables which lead to an algebraic form of the transformed Hamiltonian h , (iii) finally, exploiting the ambiguity in the definition of polynomial invariants of the fixed degrees we search for variables for which the flag of invariant subspaces of (1) is minimal flag. A primary goal of this paper is to show that the same strategy can be applied for a study of the rational H_3 Hamiltonian (1). We find the algebraic form of the rational H_3 Hamiltonian and a minimal flag of its invariant subspaces. Furthermore, we demonstrate that any (invariant) subspace from the minimal flag is a representation space of a finite-dimensional representation of a certain infinite-dimensional, finitely-generated algebra. This algebra is the hidden algebra of the H_3 system. A similar analysis is done for one of the integrals of the H_3 system.

Another goal of the paper is to find a quasi-exactly-solvable generalization of the H_3 rational model. By definition a linear differential operator is quasi-exactly-solvable (QES) if it preserves a finite-dimensional functional space with an explicitly indicated basis (see e.g.

[13]). Thus, it implies that the QES operator has a finite-dimensional invariant subspace spanned by known functions. Furthermore, one can indicate explicitly a basis where the operator being written in the matrix form has a block-triangular form. In practice, for all known examples of the QES operators the finite-dimensional invariant subspace is a space of inhomogeneous polynomials in one or several variables. In many cases the space of polynomials can be identified with a finite dimensional representation space of a Lie algebra of differential operators of the first order. In the case of crystallographic root systems a certain QES generalization has been found for each particular rational Hamiltonian [14]. All those examples are related with the existence of the hidden $sl(2)$ algebra. We show that a similar $sl(2)$ -quasi-exactly-solvable generalization of the H_3 model exists.

Finally, we show that in the H_3 orbit space (in the space of the H_3 invariants $t_a^{(\Omega)}$) there exists a discrete model defined on three-dimensional uniform lattice with polynomial eigenfunctions which is isospectral to the rational H_3 model and integrable. We will call this model the "discrete H_3 rational model".

II. THE HAMILTONIAN

The Hamiltonian of the rational H_3 model (see (1)) is invariant wrt the H_3 Coxeter group, which is the full symmetry group of the icosahedron. This discrete group is subgroup of $O(3)$ and its dimension is 120 (see e.g. [15]). In the Cartesian coordinates x_1, x_2, x_3 the Hamiltonian has the form

$$\begin{aligned} \mathcal{H}_{H_3} = & \frac{1}{2} \sum_{k=1}^3 \left[-\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2 + \frac{g}{x_k^2} \right] \\ & + \sum_{\{i,j,k\}} \sum_{\mu_{1,2}=0,1} \frac{2g}{[x_i + (-1)^{\mu_1} \varphi_+ x_j + (-1)^{\mu_2} \varphi_- x_k]^2}, \quad x \in \mathbf{R}^3 \end{aligned} \quad (6)$$

where $\{i, j, k\} = \{1, 2, 3\}$ and its even permutations. Here $g = \nu(\nu - 1) > -1/4$ is the coupling constant, $\varphi_{\pm} = (1 \pm \sqrt{5})/2$ the *golden ratio* and its algebraic conjugate. We choose as the configuration space the fundamental domain of the H_3 group – the space bounded by three planes

$$x_1 = 0, \quad x_3 = 0 \quad \text{and} \quad x_3 + \varphi_+ x_1 + \varphi_- x_2 = 0 \quad (7)$$

for $x_1, x_2, x_3 \geq 0$.

The ground state eigenfunction and its eigenvalue are

$$\Psi_0(x) = \Delta_1^\nu \Delta_2^\nu \exp \left(-\frac{\omega}{2} \sum_{k=1}^3 x_k^2 \right), \quad E_0 = \frac{3}{2} \omega (1 + 10\nu), \quad (8)$$

where

$$\Delta_1 = \prod_{k=1}^3 x_k, \quad (9)$$

$$\Delta_2 = \prod_{\{i,j,k\}} \prod_{\mu's=0,1} [x_i + (-1)^{\mu_1} \varphi_+ x_j + (-1)^{\mu_2} \varphi_- x_k]. \quad (10)$$

The ground state eigenfunction (8) does not vanish in the configuration space (7).

The main object of our study is the gauge-rotated Hamiltonian (6) with the ground state eigenfunction (8) taken as a factor,

$$h_{H_3} = -2(\Psi_0)^{-1}(\mathcal{H}_{H_3} - E_0)(\Psi_0), \quad (11)$$

where E_0 is given by (8). The gauge rotated operator (11) is the second-order differential operator without free term. By construction its lowest eigenfunction is a constant and the lowest eigenvalue is equal to zero. Now let us introduce new variables in (11).

The H_3 root space is characterized by three fundamental weights w_c , $c = 1, 2, 3$ (see e.g. [16]). Taking action of all group elements on fundamental weight ω_c we generate orbit Ω_c of a certain length (length \equiv #elements of the orbit). The results are summarized as

weight	orbit length
$w_1 = (0, \varphi_+, 1)$	12
$w_2 = (1, \varphi_+^2, 0)$	20
$w_3 = (0, 2\varphi_+, 0)$	30

In order to find H_3 -invariants (5) we choose for simplicity the shortest orbit $\Omega(w_1)$ and make averaging,

$$t_a^{(\Omega)}(x) = \sum_{w \in \Omega(w_1)} (w \cdot x)^a, \quad (12)$$

where $a = 2, 6, 10$ are the degrees of the H_3 group. These invariants are defined ambiguously, up to a non-linear combination of the invariants of the lower degrees

$$\begin{aligned} t_2^{(\Omega)} &\mapsto t_2^{(\Omega)}, \\ t_6^{(\Omega)} &\mapsto t_6^{(\Omega)} + A (t_2^{(\Omega)})^3, \\ t_{10}^{(\Omega)} &\mapsto t_{10}^{(\Omega)} + B (t_2^{(\Omega)})^2 t_6^{(\Omega)} + C (t_2^{(\Omega)})^5, \end{aligned} \quad (13)$$

where A, B, C are parameters. Now we can make a change of variables in the gauge-rotated Hamiltonian (11):

$$(x_1, x_2, x_3) \rightarrow (t_2^{(\Omega)}, t_6^{(\Omega)}, t_{10}^{(\Omega)}) .$$

The first observation is that the transformed Hamiltonian $h_{H_3}(t)$ (11) takes on an algebraic form for any value of the parameters A, B, C in variables t 's (13). The second observation is that for any value of the parameters A, B, C there exists a flag of invariant subspaces in polynomials of the Hamiltonian $h_{H_3}(t)$. Our goal is to find the parameters for which $h_{H_3}(t)$ preserves the minimal flag. After some analysis we found such a set of parameters

$$A = -\frac{13}{10}, \quad B = -\frac{76}{15}, \quad C = \frac{1531}{375} . \quad (14)$$

The t -variables (13) for such values of parameters, which we denote as τ -variables, are

$$\begin{aligned} \tau_1 &= x_1^2 + x_2^2 + x_3^2, \\ \tau_2 &= -\frac{3}{10}(x_1^6 + x_2^6 + x_3^6) + \frac{3}{10}(2 - 5\varphi_+)(x_1^2x_2^4 + x_2^2x_3^4 + x_3^2x_1^4) \\ &\quad + \frac{3}{10}(2 - 5\varphi_-)(x_1^2x_3^4 + x_2^2x_1^4 + x_3^2x_2^4) - \frac{39}{5}(x_1^2x_2^2x_3^2), \\ \tau_3 &= \frac{2}{125}(x_1^{10} + x_2^{10} + x_3^{10}) + \frac{2}{25}(1 + 5\varphi_-)(x_1^8x_2^2 + x_2^8x_3^2 + x_3^8x_1^2) \\ &\quad + \frac{2}{25}(1 + 5\varphi_+)(x_1^8x_3^2 + x_2^8x_1^2 + x_3^8x_2^2) + \frac{4}{25}(1 - 5\varphi_-)(x_1^6x_2^4 + x_2^6x_3^4 + x_3^6x_1^4) \\ &\quad + \frac{4}{25}(1 - 5\varphi_+)(x_1^6x_3^4 + x_2^6x_1^4 + x_3^6x_2^4) - \frac{112}{25}(x_1^6x_2^2x_3^2 + x_2^6x_3^2x_1^2 + x_3^6x_1^2x_2^2) \\ &\quad + \frac{212}{25}(x_1^2x_2^4x_3^4 + x_2^2x_3^4x_1^4 + x_3^2x_1^4x_2^4). \end{aligned} \quad (15)$$

The Hamiltonian $h_{H_3}(\tau)$ has infinitely-many finite-dimensional invariant subspaces

$$\mathcal{P}_n^{(1,2,3)} = \langle \tau_1^{p_1} \tau_2^{p_2} \tau_3^{p_3} \mid 0 \leq p_1 + 2p_2 + 3p_3 \leq n \rangle, \quad n = 0, 1, 2, \dots \quad (16)$$

which form the minimal flag. Its characteristic vector is

$$\vec{\alpha}_{min} = (1, 2, 3) . \quad (17)$$

It is worth noting that each particular space $\mathcal{P}_n^{(1,2,3)}$ (16) as well as the whole flag are invariant with respect to a weighted projective transformation

$$\begin{aligned} \tau_1 &\rightarrow \tau_1 + a, \\ \tau_2 &\rightarrow \tau_2 + b_1 \tau_1^2 + b_2 \tau_1 + b_3, \\ \tau_3 &\rightarrow \tau_3 + c_1 \tau_1 \tau_2 + c_2 \tau_1^3 + c_3 \tau_2 + c_4 \tau_1^2 + c_5 \tau_1 + c_6. \end{aligned} \quad (18)$$

where $\{a, b, c\}$ are parameters. It manifests a hidden invariance of the Hamiltonian (6). It is seen in a clear way in the space of orbits only.

Finally, the gauge-rotated Hamiltonian (11) in the τ -coordinates is written as

$$h_{H_3} = \sum_{i,j=1}^3 A_{ij}(\tau) \frac{\partial^2}{\partial \tau_i \partial \tau_j} + \sum_{i=1}^3 B_i(\tau) \frac{\partial}{\partial \tau_i}, \quad A_{ij} = A_{ji}, \quad (19)$$

with the coefficient functions

$$\begin{aligned} A_{11} &= 4\tau_1, \\ A_{12} &= 12\tau_2, \\ A_{13} &= 20\tau_3, \\ A_{22} &= -\frac{48}{5}\tau_1^2\tau_2 + \frac{45}{2}\tau_3, \\ A_{23} &= \frac{16}{15}\tau_1\tau_2^2 - 24\tau_1^2\tau_3, \\ A_{33} &= -\frac{64}{3}\tau_1\tau_2\tau_3 + \frac{128}{45}\tau_2^3, \\ B_1 &= 6(1 + 10\nu) - 4\omega\tau_1, \\ B_2 &= -\frac{48}{5}(1 + 5\nu)\tau_1^2 - 12\omega\tau_2, \\ B_3 &= -\frac{64}{15}(2 + 5\nu)\tau_1\tau_2 - 20\omega\tau_3. \end{aligned} \quad (20)$$

It can be easily checked that the operator (19) is triangular with respect to action on monomials $\tau_1^{p_1}\tau_2^{p_2}\tau_3^{p_3}$. One can find the spectrum of (19) $h_{H_3}\varphi = -2\epsilon\varphi$ explicitly

$$\epsilon_{n_1, n_2, n_3} = 2\omega(n_1 + 3n_2 + 5n_3), \quad (21)$$

where $n_i = 0, 1, 2, \dots$. Degeneracy of the spectrum is related to the number of solutions of the equation $n_1 + 3n_2 + 5n_3 = k$ for $k = 0, 1, 2, \dots$ in non-negative numbers $n_{1,2,3}$. The spectrum ϵ does not depend on the coupling constant g and it is equidistant. It coincides to the spectrum of 3D anisotropic harmonic oscillator with frequencies $(2\omega, 6\omega, 10\omega)$. The energies of the original rational H_3 Hamiltonian (6) are $E = E_0 + \epsilon$.

The boundary of the configuration space of the rational H_3 model (6) in the τ variables is determined by the zeros of the ground state eigenfunction, hence, by pre-exponential factor in (8). It is the algebraic surface of degree 15 in Cartesian coordinates being a product of monomials. In τ -coordinates it can be written as

$$12960\tau_1^5\tau_3^2 - 5760\tau_1^4\tau_2^2\tau_3 + 640\tau_1^3\tau_2^4 + 54000\tau_1^2\tau_2\tau_3^2 - 21600\tau_1\tau_2^3\tau_3 + 2304\tau_2^5 + 50625\tau_3^3 = 0, \quad (22)$$

which is the algebraic surface of degree seven; the equation contains monomials of the degrees 7, 5 and 3. It is worth mentioning that l.h.s. of (22) is proportional to the square of Jacobian, $J^2(\frac{\partial \tau}{\partial x})$.

III. INTEGRAL

The Hamiltonian (6) being written in spherical coordinates (r, θ, ϕ) takes a very simple form

$$\mathcal{H}_{H_3} = -\frac{1}{2}\Delta^{(3)} + \frac{1}{2}\omega^2 r^2 + \frac{W(\theta, \phi)}{r^2}, \quad (23)$$

where $\Delta^{(3)}$ is the 3D Laplacian and the angular function

$$\begin{aligned} W(\theta, \phi) = & \frac{2\nu(\nu-1)}{(s_\theta c_\phi + \varphi_+ s_\theta s_\phi + \varphi_- c_\theta)^2} + \frac{2\nu(\nu-1)}{(s_\theta c_\phi - \varphi_+ s_\theta s_\phi + \varphi_- c_\theta)^2} \\ & + \frac{2\nu(\nu-1)}{(s_\theta c_\phi + \varphi_+ s_\theta s_\phi - \varphi_- c_\theta)^2} + \frac{2\nu(\nu-1)}{(s_\theta c_\phi - \varphi_+ s_\theta s_\phi - \varphi_- c_\theta)^2} \\ & + \frac{2\nu(\nu-1)}{(s_\theta s_\phi + \varphi_+ c_\theta + \varphi_- s_\theta c_\phi)^2} + \frac{2\nu(\nu-1)}{(s_\theta s_\phi - \varphi_+ c_\theta + \varphi_- s_\theta c_\phi)^2} \\ & + \frac{2\nu(\nu-1)}{(s_\theta s_\phi + \varphi_+ c_\theta - \varphi_- s_\theta c_\phi)^2} + \frac{2\nu(\nu-1)}{(s_\theta s_\phi - \varphi_+ c_\theta - \varphi_- s_\theta c_\phi)^2} \\ & + \frac{2\nu(\nu-1)}{(c_\theta + \varphi_+ s_\theta c_\phi + \varphi_- s_\theta s_\phi)^2} + \frac{2\nu(\nu-1)}{(c_\theta - \varphi_+ s_\theta c_\phi + \varphi_- s_\theta s_\phi)^2} \\ & + \frac{2\nu(\nu-1)}{(c_\theta + \varphi_+ s_\theta c_\phi - \varphi_- s_\theta s_\phi)^2} + \frac{2\nu(\nu-1)}{(c_\theta - \varphi_+ s_\theta c_\phi - \varphi_- s_\theta s_\phi)^2} \\ & + \frac{\nu(\nu-1)}{2s_\theta^2 c_\phi^2} + \frac{\nu(\nu-1)}{2s_\theta^2 s_\phi^2} + \frac{\nu(\nu-1)}{2c_\phi^2}. \end{aligned} \quad (24)$$

Here, for the sake of simplicity we denoted $c_\vartheta \equiv \cos \vartheta$, $s_\vartheta \equiv \sin \vartheta$. It is seen immediately, that the Schroedinger equation (23) admits a separation of radial variable r : any solution can be written in factorized form

$$\Psi(r, \theta, \phi) = R(r)Q(\theta, \phi). \quad (25)$$

Functions R and Q are the solutions of the equations

$$\left[-\frac{1}{2r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{2}\omega^2 r^2 + \frac{\gamma}{r^2} \right] R(r) = E R(r), \quad (26)$$

$$\mathcal{F} Q(\theta, \phi) = \gamma Q(\theta, \phi), \quad (27)$$

respectively, while γ is the constant of separation. The operator \mathcal{F} has the form

$$\mathcal{F} = \frac{1}{2} \mathcal{L}^2 + W(\theta, \phi) , \quad (28)$$

where \mathcal{L} is the angular momentum operator:

$$\mathcal{L}^2 = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} .$$

It can be immediately checked that the Hamiltonian \mathcal{H}_{H_3} and \mathcal{F} commute,

$$[\mathcal{H}_{H_3}, \mathcal{F}] = 0 . \quad (29)$$

Hence, \mathcal{F} is an integral of motion. Thus, it has common eigenfunctions with the Hamiltonian \mathcal{H}_{H_3} .

Let us make a gauge rotation of the operator \mathcal{F} (28) with the ground state function Ψ_0 as a gauge factor,

$$f = (\Psi_0)^{-1} (\mathcal{F} - \gamma_0) \Psi_0 , \quad \gamma_0 = \frac{15}{2} \nu (1 + 15\nu) , \quad (30)$$

where γ_0 is the lowest eigenvalue of \mathcal{F} , and make a change of variables to the τ variables (15). The operator f has an algebraic form,

$$f = \sum_{i,j=1}^3 F_{ij} \frac{\partial^2}{\partial \tau_i \partial \tau_j} + \sum_{j=1}^3 G_j \frac{\partial}{\partial \tau_j} , \quad F_{ij} = F_{ji} \quad (31)$$

where

$$\begin{aligned} F_{11} &= 0 , \\ F_{12} &= 0 , \\ F_{13} &= 0 , \\ F_{22} &= \frac{24}{5} \tau_1^3 \tau_2 - \frac{45}{4} \tau_1 \tau_3 + 18 \tau_2^2 , \\ F_{23} &= -\frac{8}{15} \tau_1^2 \tau_2^2 + 12 \tau_1^3 \tau_3 + 30 \tau_2 \tau_3 , \\ F_{33} &= -\frac{64}{45} \tau_1 \tau_2^3 + \frac{32}{3} \tau_1^2 \tau_2 \tau_3 + 50 \tau_3^2 , \\ G_1 &= 0 , \\ G_2 &= \frac{24}{5} (1 + 5\nu) \tau_1^3 + 3(7 + 30\nu) \tau_2 , \\ G_3 &= \frac{32}{15} (2 + 5\nu) \tau_1^2 \tau_2 + 5(11 + 30\nu) \tau_3 . \end{aligned} \quad (32)$$

It is worth noting that in the operator f the variable τ_1 appears as a parameter. It implies that any eigenfunction of h_{H_3} which depends on τ_1 only (see below Ch.V for a discussion) is the eigenfunction of the integral f with zero eigenvalue.

It can be also shown that the operator f has infinitely many finite-dimensional invariant subspaces in polynomials

$$\mathcal{P}_n^{(1,3,5)} = \langle \tau_1^{p_1} \tau_2^{p_2} \tau_3^{p_3} \mid 0 \leq p_1 + 3p_2 + 5p_3 \leq n \rangle, \quad n = 0, 1, 2, \dots \quad (33)$$

which form a flag with characteristic vector $(1, 3, 5)$.

The spectrum of the integral \mathcal{F} can be found in a closed form,

$$\gamma_{0,k_2,k_3} = 2(3k_2 + 5k_3)^2 - 30k_2k_3 + (1 + 30\nu)(3k_2 + 5k_3) + \gamma_0, \quad (34)$$

where $k_2, k_3 = 0, 1, 2, \dots$ and γ_0 is given by (29).

It can be shown that the Hamiltonian h_{H_3} has a certain degeneracy – it preserves two different flags: one with minimal characteristic vector $(1,2,3)$ and another one with characteristic vector $(1,3,5)$. The fact that the operator h_{H_3} with coefficients (20) commutes with f given by (31) implies that common eigenfunctions of the operators h_{H_3} and f are elements of the flag of spaces $\mathcal{P}^{(1,3,5)}$.

Let us denote $\phi_{n,i}$ the common eigenfunctions of h_{H_3} and f which are elements of the invariant space $\mathcal{P}_n^{(1,3,5)}$ and their respectful eigenvalues $\epsilon_{n,i}, \gamma_{n,i}$. The index i numerates these eigenfunction for given n starting from 0. The function $\phi_{n,i}$ is related to the eigenfunction of the Hamiltonian \mathcal{H}_{H_3} (6) (and the integral \mathcal{F}) through $\Psi_{n,i} = \Psi_0 \phi_{n,i}$. Thus, the eigenfunctions $\{\phi\}$ are orthogonal with the weight factor $|\Psi_0|^2$. As an illustration let us give explicit expressions for several eigenfunctions $\phi_{n,i}$ and their respectful eigenvalues,

- $n = 0$

$$\phi_{0,0} = 1, \quad \epsilon_{0,0} = 0, \quad \gamma_{0,0} = 0.$$

- $n = 1$

$$\phi_{1,0} = \tau_1 - \frac{3}{2\omega}(1 + 10\nu), \quad \epsilon_{1,0} = 2\omega, \quad \gamma_{1,0} = 0.$$

- $n = 2$

$$\phi_{2,0} = \tau_1^2 - \frac{5}{\omega}(1 + 6\nu)\tau_1 + \frac{15}{4\omega^2}(1 + 6\nu)(1 + 10\nu), \quad \epsilon_{2,0} = 4\omega, \quad \gamma_{2,0} = 0.$$

- $n = 3$

$$\phi_{3,0} = \tau_1^3 - \frac{3}{2\omega}(7 + 30\nu)\tau_1^2 + \frac{15}{4\omega^2}(1 + 6\nu)(7 + 30\nu)\tau_1 - \frac{15}{8\omega^3}(1 + 6\nu)(7 + 30\nu)(1 + 10\nu),$$

$$\epsilon_{3,0} = 6\omega, \quad \gamma_{3,0} = 0.$$

$$\phi_{3,1} = \tau_2 + \frac{8(1 + 5\nu)}{5(7 + 30\nu)}\tau_1^3, \quad \epsilon_{3,1} = 6\omega, \quad \gamma_{3,1} = 21 + 90\nu.$$

As stated before, the Hamiltonian h_{H_3} preserves two flags with characteristic vectors $(1, 2, 3)$ and $(1, 3, 5)$, respectively. The angle between the normal vectors of the minimal flag $(1, 2, 3)$ and of the basic one $(1, 1, 1)$ is given

$$\cos \theta_h = \frac{6}{\sqrt{42}} \quad \text{or} \quad \theta_h \simeq 0.39,$$

while between the vectors $(1, 3, 5)$ and $(1, 1, 1)$

$$\cos \theta_f = \frac{9}{\sqrt{105}} \quad \text{or} \quad \theta_f \simeq 0.50.$$

It seems evident that if one or more extra integrals exist they will take an algebraic form in τ -variables after the gauge rotation.

IV. DISCRETE UNIFORM H_3 SYSTEM

The existence of the algebraic form of the H_3 Hamiltonian in the space of invariants allows us to construct a discrete system with a remarkable property of isospectrality. This construction is based on employment of a quantum canonical transformation as a basis to perform a discretization of a continuous system [17]. Such a procedure was called a *Lie-algebraic discretization*. It was already used in the past to construct the isospectral discrete model of the harmonic oscillator (the A_1 system in the Hamiltonian Reduction nomenclature) in the space of $\tau = x^2$ [18].

Let us introduce a set of the finite-difference operators

$$\begin{aligned} \mathcal{D}_i^{(\delta_i)} f(\tau_i) &\equiv \frac{f(\tau_i + \delta_i) - f(\tau_i)}{\delta_i} = \frac{(e^{\delta_i \partial_i} - 1)}{\delta_i} f(x_i), \\ \mathcal{X}_i^{(\delta_i)} f(\tau_i) &\equiv \tau_i f(\tau_i - \delta_i) = (\tau_i e^{-\delta_i \partial_i}) f(\tau_i), \end{aligned} \tag{35}$$

where δ_i , $i = 1, 2, 3$ are spacings; here no summation over repeated indexes is implied. The operator $\mathcal{D}_i^{(\delta_i)}$ is the finite-difference derivative or discrete momentum; sometimes, it is

called the Norlund derivative. The operator $\mathcal{X}_i^{(\delta_i)}$ is a discrete analogue of the multiplication operator. The operators $\mathcal{D}_i^{(\delta_i)}$ and $\mathcal{X}_i^{(\delta_i)}$ form a canonical pair,

$$[\mathcal{D}_i^{(\delta_i)}, \mathcal{D}_j^{(\delta_j)}] = 0, [\mathcal{X}_i^{(\delta_i)}, \mathcal{X}_j^{(\delta_j)}] = 0, [\mathcal{D}_i^{(\delta_i)}, \mathcal{X}_j^{(\delta_j)}] = \delta_{ij}, \quad (36)$$

for $i, j = 1, 2, 3$. Hence, the operators (35) span the 7-dimensional Heisenberg algebra realizing a three-parametric quantum canonical transformation with parameters $\delta_{1,2,3}$. In the limit when all δ_i tend to zero the operators (35) gives rise to a standard coordinate-momentum representation,

$$\mathcal{D}_i^{(\delta_i)} \rightarrow \partial_i, \quad \mathcal{X}_i^{(\delta_i)} \rightarrow \tau_i.$$

Take a linear differential operator $\mathcal{L}(\partial_i, \tau_i)$. Consider the eigenvalue problem

$$\mathcal{L}(\partial_i, \tau_i) \varphi(\tau) = \lambda \varphi(\tau), \quad (37)$$

and assume it has polynomial eigenfunctions. Performing the canonical transformation (35) we arrive at

$$\mathcal{L}(\mathcal{D}_i^{(\delta_i)}, \mathcal{X}_i^{(\delta_i)}) \varphi(\mathcal{X}_i^{(\delta_i)})|0\rangle = \lambda \varphi(\mathcal{X}_i^{(\delta_i)})|0\rangle \quad (38)$$

In order to make sense to this equation one should introduce the vacuum $|0\rangle$:

$$\mathcal{D}_i^{(\delta_i)}|0\rangle = 0, \quad i = 1, 2, 3. \quad (39)$$

Then the equation (38) has a meaning of an operator eigenvalue problem in the Fock space with vacuum (39). Now let us show that the eigenvalue problem (38) has polynomial eigenfunctions and their eigenvalues are the same eigenvalues as for the polynomial eigenfunctions as the original (continuous) problem (37).

In order to exploit the representation (35) let us first define the vacuum $|0\rangle$. The condition (39) in explicit form is

$$\begin{aligned} f(\tau_1 + \delta_1, \tau_2, \tau_3) &= f(\tau_1, \tau_2, \tau_3), \\ f(\tau_1, \tau_2 + \delta_2, \tau_3) &= f(\tau_1, \tau_2, \tau_3), \\ f(\tau_1, \tau_2, \tau_3 + \delta_3) &= f(\tau_1, \tau_2, \tau_3). \end{aligned}$$

Any periodic function with periods δ_i in the coordinates τ_i is the solution of these equations; however, without loss of generality, we can make the choice

$$f(\tau_1, \tau_2, \tau_3) = 1. \quad (40)$$

Let us now define the *quasi-monomial*

$$\tau^{(n+1)} = \tau(\tau - \delta)(\tau - 2\delta) \cdots (\tau - n\delta) . \quad (41)$$

Taking into account the relation

$$(\tau e^{-\delta \partial_\tau})^n = \tau^{(n)} e^{-n\delta \partial_\tau}$$

and choosing of vacuum (40), it is easy to check that

$$\left(\mathcal{X}_i^{(\delta_i)} \right)^n |0\rangle = \tau_i^{(n)} . \quad (42)$$

Now we can relate the solutions of (38) with the solutions of (37). Let us assume that

$$\varphi(\tau) = \sum \alpha_{klm} \tau_1^k \tau_2^l \tau_3^m \quad (43)$$

is a polynomial solution of the equation (37). The canonical transformation (35) implies the replacement of τ_i by $\mathcal{X}_i^{(\delta_i)}$. Taking in account (42) we come to the conclusion that each monomial in (43) should be replaced by a quasi-monomial. Hence, the corresponding polynomial solution of (38) is

$$\tilde{\varphi}(\tau) = \sum \alpha_{klm} \tau_1^{(k)} \tau_2^{(l)} \tau_3^{(m)} , \quad (44)$$

with the same expansion coefficients α_{klm} as in (43) and the same eigenvalue.

Performing the procedure of canonical discretization (37) \rightarrow (38) for the H_3 Hamiltonian in the algebraic form h_{H_3} (19), we arrive at the following (isospectral) finite-difference operator:

$$\tilde{h}_{H_3} \equiv h_{H_3}(\mathcal{D}_i^{(\delta_i)}, \mathcal{X}_i^{(\delta_i)}) = \sum_{k_1, k_2, k_3} A_{k_1 k_2 k_3} e^{k_1 \delta_1 \partial_1 + k_2 \delta_2 \partial_2 + k_3 \delta_3 \partial_3} , \quad (45)$$

with the following non-vanishing coefficients

$$\begin{aligned} A_{0,0,0} &= -\frac{4}{\delta_1}(2 + \delta_1 \omega) \left[\frac{\tau_1}{\delta_1} + \frac{3\tau_2}{\delta_2} + \frac{5\tau_3}{\delta_3} \right] - \frac{6}{\delta_1}(1 + 10\nu) , \\ A_{1,0,0} &= \frac{2}{\delta_1} \left[\frac{2\tau_1}{\delta_1} + \frac{12\tau_2}{\delta_2} + \frac{20\tau_3}{\delta_3} + 3(1 + 10\nu) \right] , \\ A_{-1,0,0} &= \frac{4}{\delta_1^2}(1 + \delta_1 \omega) \tau_1 , \\ A_{-2,0,0} &= \frac{48}{5\delta_2} \tau_1 (\tau_1 - \delta_1) \left[\frac{2\tau_2}{\delta_2} + \frac{5\tau_3}{\delta_3} + 1 + 5\nu \right] , \\ A_{0,-1,0} &= \frac{12}{\delta_1 \delta_2} (2 + \delta_1 \omega) \tau_2 , \\ A_{0,0,-1} &= \frac{5}{2} \left[\frac{8}{\delta_1 \delta_3} (2 + \delta_1 \omega) + \frac{9}{\delta_2^2} \right] \tau_3 , \end{aligned} \quad (46)$$

$$\begin{aligned}
A_{0,-3,0} &= \frac{128}{45\delta_3^2} \tau_2 (\tau_2 - \delta_2) (\tau_2 - 2\delta_2) , \\
A_{1,-1,0} &= -\frac{24\tau_2}{\delta_1\delta_2} , \\
A_{1,0,-1} &= -\frac{40\tau_3}{\delta_1\delta_3} , \\
A_{-1,-1,0} &= -\frac{32}{15\delta_3} \tau_1 \tau_2 \left[\frac{\tau_2}{\delta_2} - \frac{20\tau_3}{\delta_3} - 5(1 + 2\nu) \right] , \\
A_{-1,-2,0} &= \frac{32}{15\delta_2\delta_3} \tau_1 \tau_2 (\tau_2 - \delta_2) , \\
A_{-1,-1,1} &= \frac{32}{15\delta_3} \tau_1 \tau_2 \left[\frac{\tau_2}{\delta_2} - \frac{10\tau_3}{\delta_3} - 5(1 + 2\nu) \right] , \\
A_{-1,-1,-1} &= -\frac{64}{3\delta_3^2} \tau_1 \tau_2 \tau_3 , \\
A_{-1,-2,1} &= -\frac{32}{15\delta_2\delta_3} \tau_1 \tau_2 (\tau_2 - \delta_2) , \\
A_{-2,1,0} &= -\frac{48}{5\delta_2} \tau_1 (\tau_1 - \delta_1) \left[\frac{\tau_2}{\delta_2} + \frac{5\tau_3}{\delta_3} + 1 + 5\nu \right] , \\
A_{-2,-1,0} &= -\frac{48}{5\delta_2^2} \tau_2 \tau_1 (\tau_1 - \delta_1) , \\
A_{-2,0,-1} &= -\frac{48}{\delta_2\delta_3} \tau_3 \tau_1 (\tau_1 - \delta_1) , \\
A_{-2,1,-1} &= \frac{48}{\delta_2\delta_3} \tau_3 \tau_1 (\tau_1 - \delta_1) , \\
A_{0,1,-1} &= -\frac{45\tau_3}{\delta_2^2} , \\
A_{0,2,-1} &= \frac{45\tau_3}{2\delta_2^2} , \\
A_{0,-3,1} &= \frac{256}{45\delta_3^2} \tau_2 (\tau_2 - \delta_2) (\tau_2 - 2\delta_2) , \\
A_{0,-3,2} &= \frac{128}{45\delta_3^2} \tau_2 (\tau_2 - \delta_2) (\tau_2 - 2\delta_2) .
\end{aligned}$$

The corresponding eigenvalue problem is

$$\sum_{k_1, k_2, k_3} A_{k_1 k_2 k_3} \varphi(\tau_1 + k_1 \delta_1, \tau_2 + k_2 \delta_2, \tau_3 + k_3 \delta_3) = -2\epsilon \varphi(\tau_1, \tau_2, \tau_3) , \quad (47)$$

which is in the explicit form

$$\begin{aligned}
& \frac{2}{\delta_1} \left[(2 + \delta_1 \omega) \left(\frac{\tau_1}{\delta_1} + \frac{3\tau_2}{\delta_2} + \frac{5\tau_3}{\delta_3} \right) - 3(1 + 10\nu) \right] \varphi(\tau_1, \tau_2, \tau_3) \\
& + \frac{2}{\delta_1} \left[\frac{2\tau_1}{\delta_1} + \frac{12\tau_2}{\delta_2} + \frac{20\tau_3}{\delta_3} + 3(1 + 10\nu) \right] \varphi(\tau_1 + \delta_1, \tau_2, \tau_3) \\
& + \frac{4}{\delta_1^2} (1 + \delta_1 \omega) \tau_1 \varphi(\tau_1 - \delta_1, \tau_2, \tau_3) + \frac{12}{\delta_1 \delta_2} (2 + \delta_1 \omega) \tau_2 \varphi(\tau_1, \tau_2 - \delta_2, \tau_3)
\end{aligned} \quad (48)$$

$$\begin{aligned}
& + \frac{48}{5\delta_2} \tau_1 (\tau_1 - \delta_1) \left[\frac{2\tau_2}{\delta_2} + \frac{5\tau_3}{\delta_3} + (1 + 5\nu) \right] \varphi(\tau_1 - 2\delta_1, \tau_2, \tau_3) \\
& + \frac{5}{2} \left[\frac{8}{\delta_1 \delta_3} (2 + \delta_1 \omega) + \frac{9}{\delta_2^2} \right] \tau_3 \varphi(\tau_1, \tau_2, \tau_3 - \delta_3) - \frac{24\tau_2}{\delta_1 \delta_2} \varphi(\tau_1 + \delta_1, \tau_2 - \delta_2, \tau_3) \\
& \frac{128}{45\delta_3^2} \tau_2 (\tau_2 - \delta_2) (\tau_2 - 2\delta_2) \varphi(\tau_1, \tau_2 - 3\delta_2, \tau_3) - \frac{40\tau_3}{\delta_1 \delta_3} \varphi(\tau_1 + \delta_1, \tau_2, \tau_3 - \delta_3) \\
& - \frac{32}{15\delta_3} \tau_1 \tau_2 \left[\frac{\tau_2}{\delta_2} - \frac{20\tau_3}{\delta_3} - 5(1 + 2\nu) \right] \varphi(\tau_1 - \delta_1, \tau_2 - \delta_2, \tau_3) \\
& + \frac{32}{15\delta_2 \delta_3} \tau_1 \tau_2 (\tau_2 - \delta_2) \varphi(\tau_1 - \delta_1, \tau_2 - 2\delta_2, \tau_3) \\
& + \frac{32}{15\delta_3} \tau_1 \tau_2 \left[\frac{\tau_2}{\delta_2} - \frac{10\tau_3}{\delta_3} - 5(1 + 2\nu) \right] \varphi(\tau_1 - \delta_1, \tau_2 - \delta_2, \tau_3 + \delta_3) \\
& - \frac{64}{3\delta_3^2} \tau_1 \tau_2 \tau_3 \varphi(\tau_1 - \delta_1, \tau_2 - \delta_2, \tau_3 - \delta_3) \\
& - \frac{32}{15\delta_2 \delta_3} \tau_1 \tau_2 (\tau_2 - \delta_2) \varphi(\tau_1 - \delta_1, \tau_2 - 2\delta_2, \tau_3 + \delta_3) \\
& - \frac{48}{5\delta_2} \tau_1 (\tau_1 - \delta_1) \left[\frac{\tau_2}{\delta_2} + \frac{5\tau_3}{\delta_3} + (1 + 5\nu) \right] \varphi(\tau_1 - 2\delta_1, \tau_2 + \delta_2, \tau_3) \\
& - \frac{32}{15\delta_2 \delta_3} \tau_1 \tau_2 (\tau_2 - \delta_2) \varphi(\tau_1 - \delta_1, \tau_2 - 2\delta_2, \tau_3 + \delta_3) \\
& - \frac{48}{5\delta_2} \tau_1 (\tau_1 - \delta_1) \left[\frac{\tau_2}{\delta_2} + \frac{5\tau_3}{\delta_3} + (1 + 5\nu) \right] \varphi(\tau_1 - 2\delta_1, \tau_2 + \delta_2, \tau_3) \\
& - \frac{48}{5\delta_2^2} \tau_2 \tau_1 (\tau_1 - \delta_1) \varphi(\tau_1 - 2\delta_1, \tau_2 - \delta_2, \tau_3) \\
& - \frac{48}{\delta_2 \delta_3} \tau_3 \tau_1 (\tau_1 - \delta_1) \varphi(\tau_1 - 2\delta_1, \tau_2, \tau_3 - \delta_3) \\
& + \frac{48}{\delta_2 \delta_3} \tau_3 \tau_1 (\tau_1 - \delta_1) \varphi(\tau_1 - 2\delta_1, \tau_2 + \delta_2, \tau_3 - \delta_3) \\
& - \frac{45\tau_3}{\delta_2^2} \varphi(\tau_1, \tau_2 + \delta_2, \tau_3 - \delta_3) + \frac{45\tau_3}{2\delta_2^2} \varphi(\tau_1, \tau_2 + 2\delta_2, \tau_3 - \delta_3) \\
& + \frac{256}{45\delta_3^2} \tau_2 (\tau_2 - \delta_2) (\tau_2 - 2\delta_2) \varphi(\tau_1, \tau_2 - 3\delta_2, \tau_3 + \delta_3) \\
& + \frac{128}{45\delta_3^2} \tau_2 (\tau_2 - \delta_2) (\tau_2 - 2\delta_2) \varphi(\tau_1, \tau_2 - 3\delta_2, \tau_3 + 2\delta_3) = -2\epsilon \varphi(\tau_1, \tau_2, \tau_3) .
\end{aligned}$$

It defines the discrete uniform H_3 system. It is worth noting that, although we started from a second-order differential operator, the 22-point finite-difference operator occurs – it connects the function in 22 different points in the lattice space: four points in τ_1 -direction, six points in τ_2 -direction, four points in τ_3 -direction. The structure of the operator is shown in Fig.1.

The spectrum of the discrete operator \tilde{h}_{H_3} (45) for polynomial eigenfunctions coincides

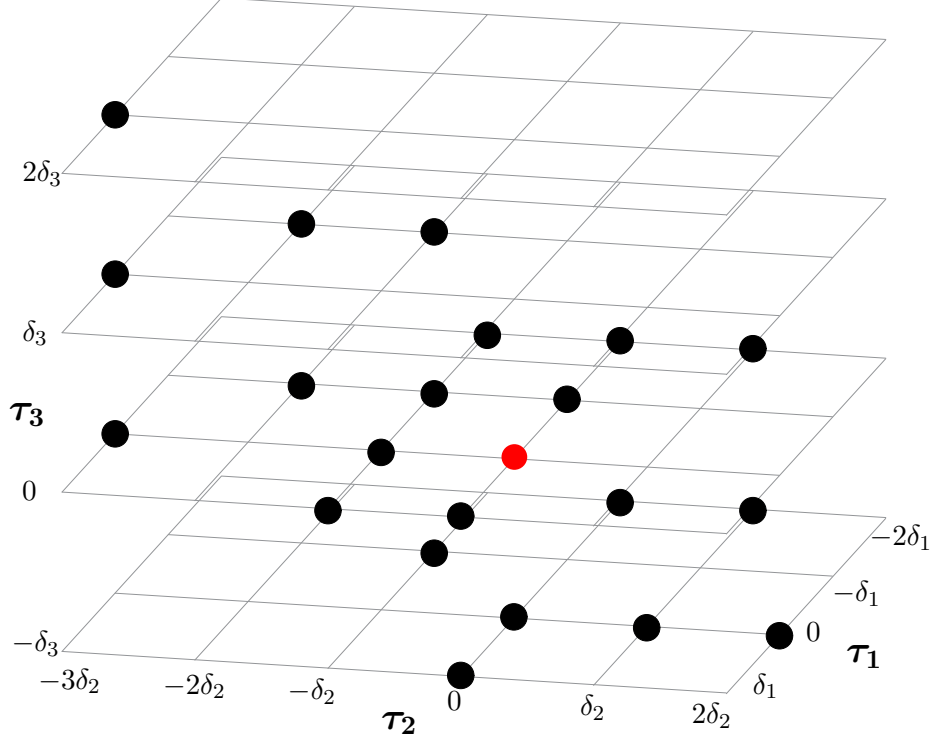


FIG. 1: Graphical representation of the 22-point discrete operator (45). The origin is shown in red.

to the spectrum of the continuous operator h_{H_3} (19) where all eigenfunctions are polynomial ones:

$$\epsilon_{n_1, n_2, n_3} = 2\omega(n_1 + 3n_2 + 5n_3) , \quad (49)$$

where the n 's are non-negative integers. The eigenfunctions of (45) are related to the eigenfunctions of the continuous operator h_{H_3} by replacing each monomial with a quasi-monomial in each variable. Such a phenomenon can be called *partial* or *polynomial isospectrality*. We cannot exclude the existence of other eigenstates of the discrete operator \tilde{h}_{H_3} than those given by polynomial eigenfunctions. These eigenstates can correspond to non-normalizable eigenfunctions of h_{H_3} .

As a particular case let us consider the unit spacing $\delta_1 = \delta_2 = \delta_3 = 1$. This case corresponds to the discretization in a cubic lattice the space of orbits (in τ -space) with unit lattice vector. The equation (48) is reduced to the equation

$$\begin{aligned} & 2[(2 + \omega)(\tau_1 + 3\tau_2 + 5\tau_3) - 3(1 + 10\nu)] \varphi(\tau_1, \tau_2, \tau_3) \\ & + 2[2\tau_1 + 12\tau_2 + 20\tau_3 + 3(1 + 10\nu)] \varphi(\tau_1 + 1, \tau_2, \tau_3) \end{aligned}$$

$$\begin{aligned}
& + 4(1 + \omega)\tau_1 \varphi(\tau_1 - 1, \tau_2, \tau_3) + 12(2 + \omega)\tau_2 \varphi(\tau_1, \tau_2 - 1, \tau_3) \\
& + \frac{5}{2}(25 + 8\omega)\tau_3 \varphi(\tau_1, \tau_2, \tau_3 - 1) - 24\tau_2 \varphi(\tau_1 + 1, \tau_2 - 1, \tau_3) \\
& + \frac{48}{5}\tau_1(\tau_1 - 1) [2\tau_2 + 5\tau_3 + (1 + 5\nu)] \varphi(\tau_1 - 2, \tau_2, \tau_3) \\
& - \frac{128}{45}\tau_2(\tau_2 - 1)(\tau_2 - 2) \varphi(\tau_1, \tau_2 - 3, \tau_3) - 40\tau_3 \varphi(\tau_1 + 1, \tau_2, \tau_3 - 1) \\
& - \frac{32}{15}\tau_1\tau_2 [\tau_2 - 20\tau_3 - 5(1 + 2\nu)] \varphi(\tau_1 - 1, \tau_2 - 1, \tau_3) \\
& + \frac{32}{15}\tau_1\tau_2(\tau_2 - 1) \varphi(\tau_1 - 1, \tau_2 - 2, \tau_3) - \frac{48}{5}\tau_2\tau_1(\tau_1 - 1) \varphi(\tau_1 - 2, \tau_2 - 1, \tau_3) \\
& + \frac{32}{15}\tau_1\tau_2 [\tau_2 - 10\tau_3 - 5(1 + 2\nu)] \varphi(\tau_1 - 1, \tau_2 - 1, \tau_3 + 1) \\
& - \frac{64}{3}\tau_1\tau_2\tau_3 \varphi(\tau_1 - 1, \tau_2 - 1, \tau_3 - 1) - \frac{32}{15}\tau_1\tau_2(\tau_2 - 1) \varphi(\tau_1 - 1, \tau_2 - 2, \tau_3 + 1) \\
& - \frac{48}{5}\tau_1(\tau_1 - 1) [\tau_2 + 5\tau_3 + (1 + 5\nu)] \varphi(\tau_1 - 2, \tau_2 + 1, \tau_3) \\
& - 48\tau_3\tau_1(\tau_1 - 1) \varphi(\tau_1 - 2, \tau_2, \tau_3 - 1) + 48\tau_3\tau_1(\tau_1 - 1) \varphi(\tau_1 - 2, \tau_2 + 1, \tau_3 - 1) \\
& - 45\tau_3 \varphi(\tau_1, \tau_2 + 1, \tau_3 - 1) + \frac{45\tau_3}{2} \varphi(\tau_1, \tau_2 + 2, \tau_3 - 1) \\
& + \frac{256}{45}\tau_2(\tau_2 - 1)(\tau_2 - 2) \varphi(\tau_1, \tau_2 - 3, \tau_3 + 1) + \frac{128}{45}\tau_2(\tau_2 - 1)(\tau_2 - 2) \varphi(\tau_1, \tau_2 - 3, \tau_3 + 2) \\
& + \frac{128}{45}\tau_2(\tau_2 - 1)(\tau_2 - 2) \varphi(\tau_1, \tau_2 - 3, \tau_3 + 2) = -2\epsilon \varphi(\tau_1, \tau_2, \tau_3) .
\end{aligned} \tag{50}$$

A similar procedure of discretization can be applied to the integral \mathcal{F} . Instead of the continuous algebraic operator f we get its discrete counterpart

$$\tilde{f} \equiv f(\mathcal{D}_i^{(\delta_i)}, \mathcal{X}_i^{(\delta_i)}) = \sum_{k_1, k_2, k_3} B_{k_1 k_2 k_3} e^{k_1 \delta_1 \partial_1 + k_2 \delta_2 \partial_2 + k_3 \delta_3 \partial_3} , \tag{51}$$

with the following coefficients

$$\begin{aligned}
B_{0, -1, 0} &= -\frac{3\tau_2}{\delta_2} \left[\frac{10\tau_3}{\delta_3} + \frac{12\tau_2}{\delta_2} + 5 + 30\nu \right] , \\
B_{0, -2, 0} &= \frac{18}{\delta_2^2} \tau_2 (\tau_2 - \delta_2) , \\
B_{0, 0, -1} &= -\frac{5\tau_3}{\delta_3} \left[\frac{6\tau_2}{\delta_2} + \frac{20\tau_3}{\delta_3} - 9 + 30\nu \right] , \\
B_{0, 0, -2} &= \frac{50}{\delta_3^2} \tau_3 (\tau_3 - \delta_3) , \\
B_{-1, 1, 0} &= \frac{12\tau_3}{\delta_2 \delta_3} \tau_1 (\tau_1 - \delta_1) (\tau_1 - 2\delta_1) ,
\end{aligned} \tag{52}$$

$$\begin{aligned}
B_{-1,1,-1} &= \frac{3}{2} \frac{\tau_1 \tau_3}{\delta_2} \left[\frac{8}{\delta_3} (\tau_1 - \delta_1)(\tau_1 - 2\delta_1) + \frac{15}{\delta_2} \right] , \\
B_{-1,2,-1} &= -\frac{45}{4\delta_2^2} \tau_1 \tau_3 , \\
B_{-1,0,-1} &= -\frac{3}{4\delta_3} \tau_1 \tau_3 \left[\frac{15}{\delta_3} - \frac{16}{\delta_2} (\tau_1 - \delta_1)(\tau_1 - 2\delta_1) \right] , \\
B_{-1,-3,0} &= -\frac{64}{45\delta_3^2} \tau_1 \tau_2 (\tau_2 - \delta_2)(\tau_2 - 2\delta_2) , \\
B_{-1,-3,1} &= \frac{128}{45\delta_3^2} \tau_1 \tau_2 (\tau_2 - \delta_2)(\tau_2 - 2\delta_2) , \\
B_{-1,-3,2} &= -\frac{64}{45\delta_3^2} \tau_1 \tau_2 (\tau_2 - \delta_2)(\tau_2 - 2\delta_2) , \\
B_{-2,-1,0} &= \frac{8}{15\delta_3} \tau_1 (\tau_1 - \delta_1) \tau_2 \left[\frac{\tau_2}{\delta_2} - \frac{40\tau_3}{\delta_3} - 9 - 20\nu \right] , \\
B_{-2,-2,0} &= -\frac{8}{15\delta_2\delta_3} \tau_1 (\tau_1 - \delta_1) \tau_2 (\tau_2 - \delta_2) , \\
B_{-2,-1,1} &= -\frac{8}{15\delta_3} \tau_1 (\tau_1 - \delta_1) \tau_2 \left[\frac{\tau_2}{\delta_2} - \frac{20\tau_3}{\delta_3} - 9 - 20\nu \right] , \\
B_{-2,-1,-1} &= \frac{32}{3\delta_3^2} \tau_1 (\tau_1 - \delta_1) \tau_2 \tau_3 , \\
B_{-2,-2,1} &= \frac{8}{15\delta_2\delta_3} \tau_1 (\tau_1 - \delta_1) \tau_2 (\tau_2 - \delta_2) , \\
B_{-3,1,0} &= \frac{24}{5\delta_2} \tau_1 (\tau_1 - \delta_1) (\tau_1 - 2\delta_1) \left[\frac{\tau_2}{\delta_2} + 1 + 5\nu \right] , \\
B_{-3,-1,0} &= \frac{24}{5\delta_2^2} \tau_1 (\tau_1 - \delta_1) (\tau_1 - 2\delta_1) \tau_2 , \\
B_{0,-1,-1} &= \frac{30}{\delta_2\delta_3} \tau_2 \tau_3 .
\end{aligned}$$

We end up with the 22-point finite-difference operator occurs – it connects the function in 22 different points in the lattice space: four points in τ_1 -direction, six points in τ_2 -direction, five points in τ_3 -direction. The structure of the operator is shown in Fig.2. It is surprising that both discrete operators \tilde{h}_{H_3} and \tilde{f} have the same structure connecting 22 points.

It is evident that the discrete operators \tilde{h}_{H_3} and \tilde{f} continue to commute. Such a procedure of discretization preserves the property of integrability.

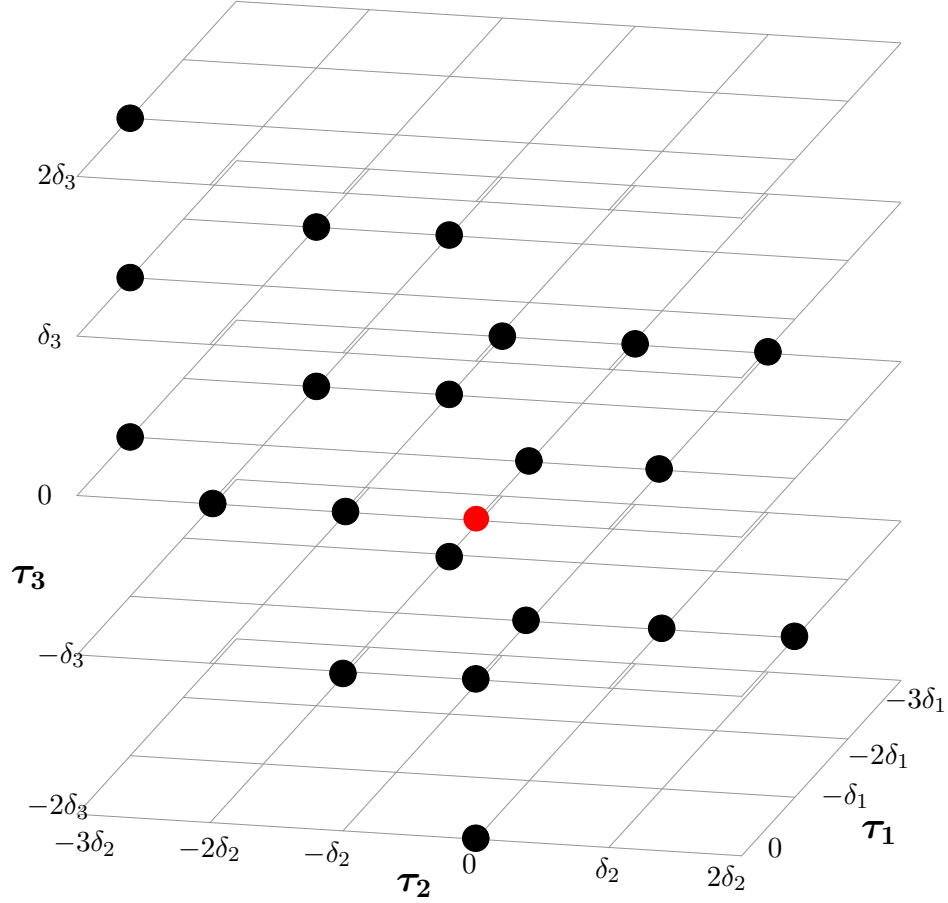


FIG. 2: Graphical representation of the 22-point discrete operator (51). The origin is shown in red.

V. QUASI-EXACTLY-SOLVABLE GENERALIZATION

Among the eigenfunctions of the Hamiltonian h_{H_3} (19) there is an infinite family of eigenfunctions depending on the single variable τ_1 . These eigenstates are solutions of the eigenvalue problem

$$-h_1 \varphi \equiv -4\tau_1 \frac{\partial^2 \varphi}{\partial \tau_1^2} + (4\omega\tau_1 - 6(1 + 10\nu)) \frac{\partial \varphi}{\partial \tau_1} = \epsilon \varphi . \quad (53)$$

Corresponding eigenfunctions are given by the Laguerre polynomials and the eigenvalues are linear in quantum number

$$\varphi_{n_1}(\tau_1) = L_{n_1}^{(1+30\nu)/2}(\omega\tau_1) , \quad \epsilon_{n_1} = 4\omega n_1 , \quad n_1 = 0, 1, 2, \dots \quad (54)$$

The operator in the l.h.s. of (53) can be rewritten in terms of the generators J_k^0, J^- of the Cartan subalgebra of the algebra $sl(2)$ of the first order differential operators:

$$J_k^+ = \tau_1^2 \frac{\partial}{\partial \tau_1} - k\tau_1, \quad J_k^0 = \tau_1 \frac{\partial}{\partial \tau_1} - \frac{k}{2}, \quad J^- = \frac{\partial}{\partial \tau_1}, \quad (55)$$

(see [14]). For integer k the generators (55) have a common invariant subspace in polynomials of degree not higher than k ,

$$\mathcal{P}_k = \langle \tau_1^p \mid 0 \leq p \leq k \rangle, \quad (56)$$

where $\dim \mathcal{P}_k = (k + 1)$. The operator (53) takes the $sl(2)$ -Lie-algebraic form

$$h_1 = 4J_0^0 J^- - 4\omega J_0^0 + 6(1 + 10\nu)J^- . \quad (57)$$

It is easy to check that the operator h_1 preserves an infinite flag of spaces of polynomials (56),

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \dots \subset \mathcal{P}_k \subset \dots, \quad (58)$$

and, in particular, any eigenfunction is an element of the flag.

Let us proceed to construct a QES generalization of (6). We look for the QES Hamiltonian in a certain form

$$H_{H_3}^{(qes)} = H_{H_3} + V^{(qes)}(\tau_1), \quad (59)$$

where $V^{(qes)}$ is a potential. Let us make a gauge rotation of (59) of the form (11). We impose the requirement that the resulting operator possesses a τ_1 -depending family of eigenfunctions. We obtain the following equation:

$$-h_1^{(qes)}\varphi \equiv -4\tau_1 \frac{\partial^2 \varphi}{\partial \tau_1^2} + (4\omega\tau_1 - 6(1 + 10\nu)) \frac{\partial \varphi}{\partial \tau_1} + 2V^{(qes)}(\tau_1)\varphi = \epsilon\varphi. \quad (60)$$

Our aim is to find $V^{(qes)}$ for which the operator $h_1^{(qes)}$ is $sl(2)$ -Lie-algebraic – can be rewritten in terms of the generators (55). Following [14], let us gauge rotate the operator (60),

$$\begin{aligned} h_1^{(sl(2)-qes)} &= \tau_1^{-\gamma} \exp\left(\frac{a}{4}\tau_1^2\right) h_1^{(qes)} \tau_1^\gamma \exp\left(-\frac{a}{4}\tau_1^2\right) \\ &= 4\tau_1 \frac{\partial^2}{\partial \tau_1^2} - 2(2a\tau_1^2 + 2\omega\tau_1 - 3 - 4\gamma - 30\nu) \frac{\partial}{\partial \tau_1} + a^2\tau_1^3 + 2a\omega\tau_1^2 \\ &\quad - 2a\left(2\gamma + 15\nu + \frac{5}{2}\right)\tau_1 + \frac{2\gamma(2\gamma + 30\nu + 1)}{\tau_1} - 4\omega\gamma - 2V^{(qes)}(\tau_1). \end{aligned} \quad (61)$$

If the corresponding potential $V^{(qes)}$ is chosen of the form

$$V^{(qes)} = \frac{1}{2} a^2 \tau_1^3 + a\omega \tau_1^2 - a \left(2k + 2\gamma + 15\nu + \frac{5}{2} \right) \tau_1 + \frac{\gamma(2\gamma + 30\nu + 1)}{\tau_1}, \quad (62)$$

the operator $h_1^{(sl(2)-qes)}$ has the Lie-algebraic form

$$h_1^{(sl(2)-qes)} = 4J_k^0 J^- - 4aJ_k^+ - 4\omega J_k^0 + 2(k + 4\gamma + 3(1 + 10\nu))J^-, \quad (63)$$

where $a \geq 0$ and γ are parameters, here the constant terms are dropped off.

It can be seen that the operator $h_1^{(sl(2)-qes)}$ (see (63)) has the space \mathcal{P}_k as the invariant subspace, but it does not preserve the flag of spaces (58). Hence, (57) has $(k+1)$ polynomial eigenfunctions of the form of polynomials of the degree k ,

$$P_j^{(k)}(\tau_1) = \sum_{i=0}^k \gamma_i^{(j)} \tau_1^i, \quad j = 0, 1, 2, \dots,$$

while other eigenfunctions are not polynomials. Now we can give the final expression of the $sl(2)$ -quasi-exactly-solvable Hamiltonian associated with the root space H_3 :

$$\begin{aligned} H_{H_3}^{(qes)} &= \frac{1}{2} \sum_{k=1}^3 \left[-\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2 + \frac{g}{x_k^2} \right] \\ &+ \sum_{\{i,j,k\}} \sum_{\mu_{1,2}=0,1} \frac{2g}{[x_i + (-1)^{\mu_1} \varphi + x_j + (-1)^{\mu_2} \varphi - x_k]^2} \\ &+ \frac{1}{2} a^2 (\mathbf{x}^2)^3 + a\omega (\mathbf{x}^2)^2 - a \left(2k + 2\gamma + 15\nu + \frac{5}{2} \right) \mathbf{x}^2 \\ &+ \frac{\gamma(2\gamma + 30\nu + 1)}{\mathbf{x}^2}, \end{aligned} \quad (64)$$

where $\{i, j, k\} = \{1, 2, 3\}$ and its even permutations, and $\mathbf{x}^2 = \sum_{i=1}^3 x_i^2$. For this Hamiltonian we know $(k+1)$ eigenstates explicitly. Their eigenfunctions are of the form

$$\Psi_k(x) = \Delta_1^\nu \Delta_2^\nu (\mathbf{x}^2)^\gamma \cdot P_k(\mathbf{x}^2) e^{-\frac{\omega}{2} \mathbf{x}^2 - \frac{a}{4} (\mathbf{x}^2)^2} \quad (65)$$

where P_k is a polynomial of degree k , the coupling constant $g = \nu(\nu - 1) > -\frac{1}{4}$ and $\Delta_{1,2}$ are given by (9). It is worth presenting several P_k explicitly,

$$\begin{aligned} P_0(\mathbf{x}^2) &= 1, \quad E_0 = \frac{3}{2} \omega \left(1 + 10\nu + \frac{4}{3} \gamma \right), \\ P_{1,\pm}(\mathbf{x}^2) &= \mathbf{x}^2 + \frac{1}{2a} \left[\omega \pm \sqrt{\omega^2 + 2a(4\gamma + 3(1 + 10\nu))} \right], \\ E_{1,\pm} &= E_0 + \omega \mp \sqrt{\omega^2 + 2a(4\gamma + 3(1 + 10\nu))}. \end{aligned} \quad (66)$$

The solutions for $k = 1$ are related through analytic continuation in one of the parameters ω, a, γ, ν keeping other parameters fixed. They form two-sheeted Riemann surface.

The QES Hamiltonian (64) is integrable – the integral \mathcal{F} (28) remains to commute with the Hamiltonian.

VI. HIDDEN ALGEBRA

We have shown that the Hamiltonian in the algebraic form (19) acts on the finite-dimensional spaces of multivariate polynomials $\mathcal{P}_n^{(1,2,3)}$, $n = 0, 1, 2, \dots$ (see (16)). A goal of this Section is to show that each one of these subspaces is a representation space of an infinite-dimensional algebra of differential operators which we call $h^{(3)}$ and to study this algebra.

The algebra $h^{(3)}$ is infinite-dimensional but finitely-generated. Their generating elements can be split into two classes. The first class of generators (lowering and Cartan operators) act in $\mathcal{P}_n^{(1,2,3)}$ for any $n \in N$ and therefore they preserve the flag $\mathcal{P}^{(1,2,3)}$. The second class operators (raising operators) act on the space $\mathcal{P}_n^{(1,2,3)}$ for a certain value of n only; they do not act on a space at other n 's.

Let us introduce the following notation for the derivatives:

$$\partial_i \equiv \frac{\partial}{\partial \tau_i}, \quad \partial_{ij} \equiv \frac{\partial^2}{\partial \tau_i \partial \tau_j}, \quad \partial_{ijk} \equiv \frac{\partial^3}{\partial \tau_i \partial \tau_j \partial \tau_k}.$$

The first class of generating elements consist of the 22 generators where 13 of them are the first order operators

$$\begin{aligned} T_0^{(1)} &= \partial_1, & T_0^{(2)} &= \partial_2, & T_0^{(3)} &= \partial_3, \\ T_1^{(1)} &= \tau_1 \partial_1, & T_2^{(2)} &= \tau_2 \partial_2, & T_3^{(3)} &= \tau_3 \partial_3, \\ T_1^{(3)} &= \tau_1 \partial_3, & T_{11}^{(3)} &= \tau_1^2 \partial_3, & T_{111}^{(3)} &= \tau_1^3 \partial_3, \\ T_1^{(2)} &= \tau_1 \partial_2, & T_{11}^{(2)} &= \tau_1^2 \partial_2, & T_2^{(3)} &= \tau_2 \partial_3, \\ & & T_{12}^{(3)} &= \tau_1 \tau_2 \partial_3, \end{aligned} \tag{67}$$

the 6 are of the second order

$$\begin{aligned} T_2^{(11)} &= \tau_2 \partial_{11}, & T_{22}^{(13)} &= \tau_2^2 \partial_{13}, & T_{222}^{(33)} &= \tau_2^3 \partial_{33}, \\ T_3^{(12)} &= \tau_3 \partial_{12}, & T_3^{(22)} &= \tau_3 \partial_{22}, & T_{13}^{(22)} &= \tau_1 \tau_3 \partial_{22}, \end{aligned} \tag{68}$$

and 2 are of the third order

$$T_3^{(111)} = \tau_3 \partial_{111} , \quad T_{33}^{(222)} = \tau_3^2 \partial_{222} . \quad (69)$$

The generators of the second class consist of 8 operators where 1 of them is of the first order

$$J_1^+ = \tau_1 J_0 , \quad (70)$$

4 are of the second order

$$J_{2,-1}^+ = \tau_2 \partial_1 J_0 , \quad J_{3,-2}^+ = \tau_3 \partial_2 J_0 , \quad J_{22,-3}^+ = \tau_2^2 \partial_3 J_0 , \quad J_2^+ = \tau_2 J_0 (J_0 + 1) , \quad (71)$$

and 3 are of the third order

$$J_{3,-11}^+ = \tau_3 \partial_{11} J_0 , \quad J_{3,-1}^+ = \tau_3 \partial_1 J_0 (J_0 + 1) , \quad J_3^+ = \tau_3 J_0 (J_0 + 1) (J_0 + 2) , \quad (72)$$

where we have introduced the diagonal operator

$$J_0 = \tau_1 \partial_1 + 2\tau_2 \partial_2 + 3\tau_3 \partial_3 - n . \quad (73)$$

for a convenience. In fact, this operator is identity operator, it is of the zeroth order and, hence, it belongs to the first class.

Before to proceed to study the commutation relations between generators we introduce a notion of conjugation. Let T_1 and T_2 be operators acting on a monomial. We say that T_2 is a conjugate to T_1 if the operator $T_2 T_1$ leaves the monomial unchanged. There is a certain ambiguity related with the central operator J_0 (73). Formally, it seems self-conjugated. From another side, the operator J_0 is, in fact, the unit operator. Thus, one can define a conjugate to J_0 to be equal to 1 and visa versa: a conjugate to 1 is equal J_0 . From this point of view any operator is defined up to a multiplicative factor of J_0 . We resolve this ambiguity by defining the generators (67)-(73) in a way for supporting below-presented commutation relations and eventually a structure of algebra. In particular, $T_0^{(1)}$ is conjugated to J_1^+ ,

$$\partial_1 \longleftrightarrow \tau_1 J_0 ,$$

and $T_0^{(3)}$ is conjugated to J_3^+ ,

$$\partial_3 \longleftrightarrow \tau_3 J_0 (J_0 + 1) (J_0 + 2) .$$

Except for this ambiguity, the conjugation coincides with the Fourier transform.

A certain number of generating operators (67)-(72) span ten Abelian subalgebras:

$$\begin{aligned}
L = \{T_0^{(3)}, T_1^{(3)}, T_{11}^{(3)}, T_{111}^{(3)}\} &\longleftrightarrow \mathfrak{L} = \{T_3^{(111)}, J_{3,-11}^+, J_{3,-1}^+, J_3^+\} \\
R = \{T_0^{(2)}, T_1^{(2)}, T_{11}^{(2)}\} &\longleftrightarrow \mathfrak{R} = \{T_2^{(11)}, J_{2,-1}^+, J_2^+\} \\
F = \{T_2^{(3)}, T_{12}^{(3)}\} &\longleftrightarrow \mathfrak{F} = \{T_3^{(12)}, J_{3,-2}^+\} \\
E = \{T_3^{(22)}, T_{13}^{(22)}\} &\longleftrightarrow \mathfrak{E} = \{T_{22}^{(13)}, J_{22,-3}^+\} \\
G = \{T_{222}^{(33)}\} &\longleftrightarrow \mathfrak{G} = \{T_{33}^{(222)}\}
\end{aligned} \tag{74}$$

The remaining generators span a non-commutative algebra isomorphic to $gl(2) \oplus \mathcal{R}^{(2)}$, where $\mathcal{R}^{(2)}$ is the Abelian algebra of dimension 2,

$$B = \{T_0^{(1)}, T_1^{(1)}, T_2^{(2)}, T_3^{(3)}, J_0, J_1^+\} \tag{75}$$

The arrow in (74) means that the corresponding operators are related by conjugation (for example, $T_0^{(3)}$ and J_3^+). The subalgebra B is the unique algebra those operators are self-conjugated.

Decomposition (74),(75) allows us to give a compact representation of the commutation relations relating the generating elements. As first we can show that the commutators between the Abelian algebras from the l.h.s. (r.h.s.) of (74) are closed – they are either zero or written in terms of generators of one of the algebras from l.h.s. (r.h.s.) plus from the algebra B

$$\begin{aligned}
[L, R] &= 0, & [\mathfrak{L}, \mathfrak{R}] &= 0, \\
[L, F] &= 0, & [\mathfrak{L}, \mathfrak{F}] &= 0, \\
[L, E] &= P_2(R), & [\mathfrak{L}, \mathfrak{E}] &= P_2(\mathfrak{R}), \\
[L, G] &= 0, & [\mathfrak{L}, \mathfrak{G}] &= 0, \\
[R, F] &= L, & [\mathfrak{R}, \mathfrak{F}] &= \mathfrak{L}, \\
[R, E] &= 0, & [\mathfrak{R}, \mathfrak{E}] &= 0, \\
[R, G] &= P_2(F), & [\mathfrak{R}, \mathfrak{G}] &= P_2(\mathfrak{F}), \\
[F, E] &= P_2(R \oplus B), & [\mathfrak{F}, \mathfrak{E}] &= P_2(\mathfrak{R} \oplus B), \\
[F, G] &= 0, & [\mathfrak{F}, \mathfrak{G}] &= 0, \\
[E, G] &= P_3(F \oplus B), & [\mathfrak{E}, \mathfrak{G}] &= P_3(\mathfrak{F} \oplus B),
\end{aligned}$$

Here $P_k(Q)$ means that the commutator is given by a polynomial of the k -th degree in the generators of Q . It turns out all commutators are symmetric under conjugation. This property also holds for cross commutators of the algebras from the l.h.s. and the r.h.s. in (74),

$$\begin{aligned}
[L, \mathfrak{R}] &= P_2(F \oplus B), & [\mathfrak{L}, R] &= P_2(\mathfrak{F} \oplus B), \\
[L, \mathfrak{F}] &= P_2(R \oplus B), & [\mathfrak{L}, F] &= P_2(\mathfrak{R} \oplus B), \\
[L, \mathfrak{E}] &= P_2(F), & [\mathfrak{L}, E] &= P_2(\mathfrak{F}), \\
[L, \mathfrak{G}] &= P_2(R \oplus E), & [\mathfrak{L}, G] &= P_2(\mathfrak{R} \oplus \mathfrak{E}), \\
[R, \mathfrak{F}] &= E, & [\mathfrak{R}, F] &= \mathfrak{E}, \\
[R, \mathfrak{E}] &= P_2(F \oplus B), & [\mathfrak{R}, E] &= P_2(\mathfrak{F} \oplus B), \\
[R, \mathfrak{G}] &= 0, & [\mathfrak{R}, G] &= 0, \\
[F, \mathfrak{E}] &= G, & [\mathfrak{F}, E] &= \mathfrak{G}, \\
[F, \mathfrak{G}] &= P_2(E \oplus B), & [\mathfrak{F}, G] &= P_2(\mathfrak{E} \oplus B), \\
[E, \mathfrak{G}] &= 0, & [\mathfrak{E}, G] &= 0,
\end{aligned}$$

The commutator between any Abelian algebra from (74) and the algebra B is of the type of a semidirect product:

$$\begin{aligned}
[L, B] &= L, & [R, B] &= R, & [F, B] &= F, & [E, B] &= E, & [G, B] &= G, \\
[\mathfrak{L}, B] &= \mathfrak{L}, & [\mathfrak{R}, B] &= \mathfrak{R}, & [\mathfrak{F}, B] &= \mathfrak{F}, & [\mathfrak{E}, B] &= \mathfrak{E}, & [\mathfrak{G}, B] &= \mathfrak{G},
\end{aligned}$$

At last, let us indicate commutators between conjugated algebras in (74) - all of them are polynomials in generators of B :

$$\begin{aligned}
[L, \mathfrak{L}] &= P_3(B), & [R, \mathfrak{R}] &= P_2(B), & [F, \mathfrak{F}] &= P_2(B), \\
[E, \mathfrak{E}] &= P_3(B), & [G, \mathfrak{G}] &= P_4(B).
\end{aligned}$$

Latter two types of relations are represented by triangular diagrams, see for example, Fig.3. In general, the commutation relations between two operators are characterized by a non-linear combination of the generators in r.h.s.. Calculating double, triple, etc. commutators one can see that the commutation relations can not be closed at any order and the order of

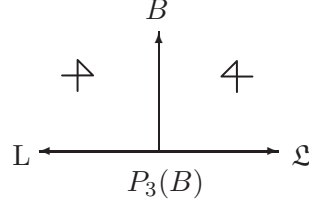


FIG. 3: Triangular diagram relating the subalgebras L , \bar{L} and B . It is a generalization of Gauss decomposition for semi-simple algebras.

the r.h.s. is increasing. Hence, the $h^{(3)}$ algebra is not a polynomial algebra. It is the infinite dimensional algebra of ordered monomials in the 30 generating operators (67)-(72) shown above.

Since $h^{(3)}$ is the algebra of differential operators acting on $\mathcal{P}_n^{(1,2,3)}$ it should be possible to write the h_{H_3} Hamiltonian (19) as a combination of the (flag-preserving) generating elements (67)-(69) of $h^{(3)}$. The $h^{(3)}$ -algebraic form of the H_3 model (6) is the following:

$$\begin{aligned}
 h_{H_3} = & 4T_1^{(1)}T_0^{(1)} + 24T_2^{(2)}T_0^{(1)} + 40T_3^{(3)}T_0^{(1)} - \frac{48}{5}T_2^{(2)}T_{11}^{(2)} + \frac{45}{2}T_3^{(22)} + \frac{32}{15}T_{12}^{(3)}T_2^{(2)} \\
 & - 48T_3^{(3)}T_{11}^{(2)} - \frac{64}{3}T_3^{(3)}T_{12}^{(3)} + \frac{128}{45}T_{222}^{(33)} + (6 + 60\nu)T_0^{(1)} - 4\omega T_1^{(1)} \\
 & - \frac{48}{5}(1 + 5\nu)T_{11}^{(2)} - 12\omega T_2^{(2)} - \frac{64}{15}(2 + 5\nu)T_{12}^{(3)} - 20\omega T_3^{(3)} .
 \end{aligned} \tag{76}$$

VII. CONCLUSIONS

We have shown that the H_3 rational system related to the non-crystallographic root system H_3 is exactly solvable with the characteristic vector $(1, 2, 3)$ [27]. This work complements the previous studies of the rational (and trigonometric) models, related with crystallographic root systems (e.g. [7] - [11]). A certain significance of exploration of the H_3 rational system is due to a fact that this model is defined in three-dimensional Euclidian (physical) space. There are very few known exactly-solvable systems in this space – the Coulomb problem, four-body Calogero-Sutherland (A_3) and BC_3 rational-trigonometric models among them. Surprisingly, all of them are integrable while the Coulomb, A_3 and BC_3 rational problems are superintegrable. The same is correct for all known two-dimensional exactly-solvable problems in the Euclidean space: all of them are integrable.

Taking Coxeter invariants of H_3 as coordinates provided us a way to reduce the rational H_3 Hamiltonian to algebraic form. It gave us a chance to find the eigenfunctions of the rational H_3 Hamiltonian which are proportional to polynomials in these invariant coordinates. It seems correct that these eigenfunctions exhaust all eigenfunctions in the Hilbert space. It is worth noting that the matrix $A_{ij}(\tau)$ which appears in front of the second derivatives after changing variables in Laplacian from Cartesian to the H_3 Coxeter invariant coordinates (see Eqs. (20)) has polynomial entries corresponding to flat space metric, hence the Riemann tensor vanishes. We call metric the *Arnold* metric [28].

It should be stressed that it was stated that the Hamiltonian of the H_3 rational system (1) is completely integrable [2]. This implies the existence of two mutually-commuted operators (the ‘higher Hamiltonians’) which commute with the Hamiltonian forming a commutative algebra. It is known (see [1]) for the crystallographic systems that these higher Hamiltonians are the differential operators of the degrees which coincide to the minimal degrees of the root space (the Lie algebra) or their squares for the A_N case. It may suggest that for the H_3 rational system the commuting integrals might be differential operators of the orders six and ten. Their explicit forms are not known so far. It is evident that these commuting operators should take on an algebraic form after a gauge rotation (with the ground state function as a gauge factor), and a change of variables from Cartesian coordinates to the Coxeter invariant variables τ ’s. Following the experience with different integrable systems, it seems the integral(s) related with separation of variables do not enter to the commutative algebra. Therefore, the integral \mathcal{F} is out of the commutative algebra of integrals. It might serve as an indication to a superintegrability of the H_3 rational system.

An analysis similar to the analysis of this paper has not yet been presented for the case of the rational systems related to the non-crystallographic root spaces H_4 . A study in progress indicates that the characteristic vector for the quantum H_4 integrable system is $(1, 5, 8, 12)$. In the case of dihedral group $I_2(k)$ the rational model has the characteristic vector $(1, k)$ [21]. It should be pointed out that unlike the rational models it is not possible to construct integrable (and exactly-solvable) trigonometric systems related to the non-crystallographic root spaces as a natural generalization of the Hamiltonian Reduction Method [1].

The existence of algebraic form of the H_3 rational Olshanetsky-Perelomov Hamiltonian makes possible the study of their polynomial perturbations which are invariant wrt the H_3 Coxeter group by purely algebraic means: one can develop a perturbation theory in which

all corrections are found by linear algebra methods [22]. In particular, it gives a chance to calculate the H_3 Coxeter-invariant, polynomial correlation functions by algebraic means.

Another important property of the existence of algebraic form of the H_3 rational Hamiltonian is a chance to perform a canonical, Lie-algebraic discretization to uniform (see Ch.IV) and exponential [23] lattices. In the case of both lattices such a discretization preserves a property of integrability, polynomiality of the eigenfunctions remains and it is isospectral. Although it does not give a hint how to introduce a scalar product for a discrete model. Making the weighted projective transformation (18) of the H_3 algebraic form (19) we arrive at different algebraic form of the H_3 Hamiltonian. Making then the Lie-algebraic discretization we arrive at a discrete model related to an original discrete model via change of variables. It can be considered as a definition of a polynomial change of variables for discrete operators.

We found the $sl(2)$ -quasi-exactly-solvable generalization of the H_3 model which remains integrable. This is the first example of quasi-exact-solvability related to non-crystallographic root systems. It complements the results obtained previously for all rational models related to crystallographic systems (see [14]) and for the $I_2(k)$ rational model [21] – each of these models admit a certain $sl(2)$ -QES generalization.

Thank to the explicit knowledge of the ground state function (8) supersymmetric H_3 model can be constructed following a procedure realized in [24] for A_N rational model, in [25] for the BC_N rational model and in [26] for the $I_2(k)$ rational model. It can be done elsewhere.

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- [27] It is worth noting that in the past there existed a single attempt to study the H_3 rational system [19]
- [28] Many years ago V.I. Arnold [20] pointed out that the contravariant flat metric on the space of orbits of any Coxeter group written in terms of the polynomial invariants has polynomial matrix elements. We have to add that for such a metric coefficient functions in front of the first derivative terms in the Laplace-Beltrami operator are also polynomials [3]